Chapter 3: Complex Analysis

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If you try to solve the equation

$$x^2 + c^2 = 0,$$

the solution is of the form

$$x = \frac{\pm\sqrt{-4c^2}}{2}.$$

Of course, this number is not real!

To try to give a solution in that problem, Euler defined the **imaginary unit**:

$$i = \sqrt{-1}.$$

Using this unit,

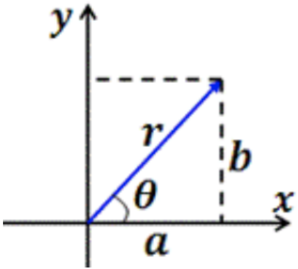
$$x = \pm i.$$

We can define the **complex field** as the collection of the complex numbers $z = a + bi$:

$$\mathbb{C} = \{z = a + bi \ : \ a, b \in \mathbb{R}\}.$$

A complex number can be written as:

- $z = a + ib$ (binomial form).
- $z = (a, b)$ (Cartesian form).
- $z = r_\alpha$ (Polar form), $r = \sqrt{a^2 + b^2}$, $\alpha = \arctan(b/a)$.



Given a complex number $z = a + bi$, we say that a and b are the real and complex part of z:

$$\mathrm{Re}(z) = a, \;\; \mathrm{Im}(z) = b.$$

We say that two complex numbers z_1, z_2 are equal if

$$\mathrm{Re}(z_1) = \mathrm{Re}(z_2), \;\; \text{and} \;\; \mathrm{Im}(z_1) = \mathrm{Im}(z_2).$$

Of course, $\mathbb{R}$ is a subset of $\mathbb{C}$ since each real number x could be written as $z = x + 0i \in \mathbb{C}$.

We say that $z \in \mathbb{C}$ is **pure imaginary** if $\mathrm{Re}(z) = 0$.

As in the real field $\mathbb{R}$, we can define the sum and product of two complex numbers: if $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$,

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2).$$

$$z_1 z_2 = (a_1 + ib_i)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2).$$

For the sum:

- Commutative: $z_1 + z_2 = z_2 + z_1$.
- Associative: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
- Neutral element: $z + 0 = 0 + z = z$, where $0 = 0 + i0$.
- Opposite element: $z + (-z) = 0$, where $-z = -a - ib$ if $z = a + bi$.

For the product:

- Commutative: $z_1 z_2 = z_2 z_1$.
- Associative: $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
- Neutral element: $z \cdot 1 = z$, where $1 = 1 + i0$.
- Reverse element: $zz^{-1} = 1$, where, if $z = a + bi$,

$$z^{-1} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}.$$

- Distributive of the product with respect to the sum:

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

The conjugate of a complex number

Given a complex number $z = a + bi$, the conjugate is

$$\overline{z} = a - bi.$$

Thanks to this number, we can see the inverse of a complex number:

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}}.$$

Then, $zz^{-1} = 1$.

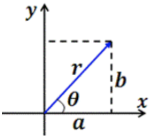
- $\text{Im}(z) = 0$ if and only if $z = \overline{z}$.
- z is pure imaginary if and only if $z = -\overline{z}$.

Modulus of a complex number

As we have said at the beginning, we can write $z = a + ib$ as (a, b), so we can have a correspondence between $\mathbb{C}$ and $\mathbb{R}^2$:

$$\mathbb{C} \to \mathbb{R}^2,$$

where for each $z = a + ib \in \mathbb{C}$, we have the correspondent vector (a, b).Also, thanks to this correspondence, we also can define the modulus as $|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$.



In the case when $za + i0$, $|z| = \sqrt{a^2} = |a|$ we recover the usual absolute value.

Properties about the modulus

- $|z| = 0$ if and only if $z = 0$.
- $|z| = |-z| = |-\overline{z}|$.
- $|zw| = |z||w|$.
- $|z^{-1}| = 1/|z|$.
- $$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}.$$
- $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$.
- $|z + w| \leq |z| + |w|$.

Given the following numbers:

$$z_1 = 1 - 2i,\ z_2 = -2 + i,\ z_3 = 3 + 5i,$$

calculate:

- $z_1 - z_3$.
- z_2^{-1}.
- $z_1\left(z_2 + \overline{z_3}\right)$.
- $\frac{z_1}{z_3}$.
- $z_3\overline{z_3}$.
- $\left|z_2\right|$.